Finite difference method for solving Advection-Diffusion Problem in 1D

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MATH 5370: Final Project

Outline

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Advection-Diffusion Problem

Background of the Advection-Diffusion Problem

- The advection-diffusion equation describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and advection.

  Advection is a transport mechanism of a substance or conserved property by a fluid due to the fluid's bulk motion.

  Diffusion is the net movement of molecules or atoms from a region of high concentration to a region of low concentration.

- The advection-diffusion equation is a relatively simple equation describing flows, or alternatively, describing a stochastically-changing system.

1D Advection-Diffusion Problem (Cont.)

General form of the 1D Advection-Diffusion Problem

The general form of the 1D advection-diffusion is given as:

$$\frac{dU}{dt} = \epsilon \frac{d^2 U}{dx^2} - a \frac{dU}{dx} + F \tag{1}$$

where,

U is the variable of interest

t is time

ϵ is the diffusion coefficient

a is the average velocity

F describes "sources" or "sinks" of the quantity U.

In Equation 1, the four terms represent the transient, diffusion, advection and source or sink term respectively.

Advection-Diffusion Problem (Cont.)

Stationary Advection-Diffusion Problem in 1D

The stationary advection-diffusion equation describes the steady-state behavior of an advection-diffusive system. In steady-state, $\frac{dU}{dt} = 0$, so Equation 1 reduces to,

$$-\epsilon \frac{d^2U}{dx^2} + a\frac{dU}{dx} = F(x). \quad (2)$$

In Equation 2, the three terms represent the diffusion, advection and source or sink term respectively.

Solution of the Stationary Advection-Diffusion Problem in 1D

Computers are often used to numerically approximate the solution of the advection-diffusion equation typically using the finite difference method (FDM) and the finite element method (FEM).

- The FDM are numerical methods for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. FDMs are thus discretization methods.
- The FEM is a numerical technique for finding approximate solutions to boundary value problems for partial differential equations. It uses subdivision of a whole problem domain into simpler parts, called finite elements, and variational methods from the calculus of variations to solve the problem by minimizing an associated error function.

Solution of the Stationary Advection-Diffusion Problem in 1D (Cont.)

Stationary Advection-Diffusion Problem in 1D

$$-\epsilon\frac{d^2U}{dx^2}+a(x)\frac{dU}{dx}=F(x),\quad 0<x<1, \tag{3}$$

$$U(0)=\alpha,\quad U(1)=\beta,\quad a(x)>a_0>0. \tag{4}$$

Where the data is chosen according to:

$$U(x)=x-\frac{\exp\left(-\frac{(1-x)}{\epsilon}\right)-\exp\left(-\frac{1}{\epsilon}\right)}{1-\exp\left(-\frac{1}{\epsilon}\right)} \tag{5}$$

$$a(x)=1 \tag{6}$$

Solution of the Stationary Advection-Diffusion Problem in 1D (Cont.)

In order to numerically solve Equation 3, we need to determine the unknown function $F(x)$ and unknown constants α and β. In order to determine the unknown constants α and β, we plug 0 and 1 into Equation 5. Thus we have,

$$U(0) = 0 - \frac{\exp\left(-\frac{(1-0)}{\epsilon}\right) - \exp\left(-\frac{1}{\epsilon}\right)}{1 - \exp\left(-\frac{1}{\epsilon}\right)} = 0 = \alpha \qquad (7)$$

and

$$U(1) = 1 - \frac{\exp\left(-\frac{(1-1)}{\epsilon}\right) - \exp\left(-\frac{1}{\epsilon}\right)}{1 - \exp\left(-\frac{1}{\epsilon}\right)} = 0 = \beta \qquad (8)$$

Solution of the Stationary Advection-Diffusion Problem in 1D (Cont.)

Similarly, in order to determine the unknown function $F(x)$ we substitute Equation 5 into Equation 3 and we get,

$$-\epsilon\left[\frac{-\exp(-\frac{1}{\epsilon})\exp(\frac{x}{\epsilon})}{\epsilon^2(1-\exp\left(-\frac{1}{\epsilon}\right))}\right]+\left[1-\frac{-\exp(-\frac{1}{\epsilon})\exp(\frac{x}{\epsilon})}{\epsilon(1-\exp\left(-\frac{1}{\epsilon}\right))}\right]=1=F(x)$$

Therefore our problem reduces to:

Stationary Advection-Diffusion Problem in 1D

$$-\epsilon\frac{d^2U}{dx^2}+\frac{dU}{dx}=1,\quad 0<x<1, \tag{9}$$

$$U(0)=0,\quad U(1)=0. \tag{10}$$

Solution of the Stationary Advection-Diffusion Problem in 1D (Cont.)

We now employ FDM to numerically solve the Stationary Advection-Diffusion Problem in 1D (Equation 9). We will employ FDM on an equally spaced grid with step-size h. We set $x_{i\pm1} = x_i \pm h$, $h = \frac{x_{n+1} - x_0}{n}$ and $x_0 = 0$, $x_{n+1} = 1$.

A finite difference method comprises a discretization of the differential equation using the grid points x_i, where the unknowns U_i (for $i = 0, \ldots, n+1$) are approximations to $U(x_i)$.

Solution of the Stationary Advection-Diffusion Problem in 1D (Cont.)

$U'(x)$ is approximated by the centered-difference:

$$(D^o U)(x) = \frac{U(x+h) - U(x-h)}{2h} \approx \frac{U_{i+1} - U_{i-1}}{2h}$$

$U''(x)$ is approximated by the central difference approximations:

$$\begin{aligned}(D^+ D^- U)(x) &= \frac{U(x+h) - 2U(x) + U(x-h)}{h^2} \\ &\approx \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\end{aligned}$$

From Equation 9, we approximate the diffusion term by the second order central-difference operator and the advection term by the centered-difference operator.

Centered-Difference method for the Stationary Advection-Diffusion Problem in 1D

Implementation of Centered-difference method for Advection-Diffusion Problem in 1D

$$-\epsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + \frac{U_{i+1} - U_{i-1}}{2h} = 1, \quad \text{on} \quad [0, 1]$$
$$U_0 = 0, \quad U_{n+1} = 0.$$

Combining terms with the same indices, we get:

$$AU_{i+1} + BU_i + CU_{i-1} = f(x_i) = 1, \tag{11}$$

where, $A = \frac{1}{2h} - \frac{\epsilon}{h^2}$, $B = \frac{2\epsilon}{h^2}$ and $C = \frac{-1}{2h} - \frac{\epsilon}{h^2}$

Centered Difference method for the Stationary Advection-Diffusion Problem in 1D (Cont.)

From Equation 11, we have a tridiagonal linear system of n equations with n unknowns, which can be written in the form

$$AU = F \tag{12}$$

where $U = [U_1, U_2, \ldots, U_n]^T$ is the unknown vector and

$$A = \begin{bmatrix} b & c & & & & 0 \\ a & b & c & & & \\ & a & b & c & & \\ & & \ddots & \ddots & \ddots & \\ & & & a & b & c \\ 0 & & & & a & b \end{bmatrix}, \quad F = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{bmatrix} \tag{13}$$

Numerical Results

A $C++$ module was developed to generate the approximated solution U_h by solving the tridiagonal system.

The tridiagonal system is solved in two steps.

- The first step is using the process of Gaussian Elimination to obtain a triangular matrix.
- The second step is using back substitution to solve for the unknown vectors.

Numerical Results (Cont.)

We will briefly present some numerical results for the advection-diffusion problem.

We will consider three(3) different cases where the number of grid points are chosen as $n = 50, 25$ and 2.
For each grid point, we will change the choice of the diffusion coefficient ϵ to $1, 0.5, 0.1$ and 0.01.

The exact solution and approximated solution are plotted on the same window for different values of ϵ on $[0, 1]$.

Numerical Results (Cont.)

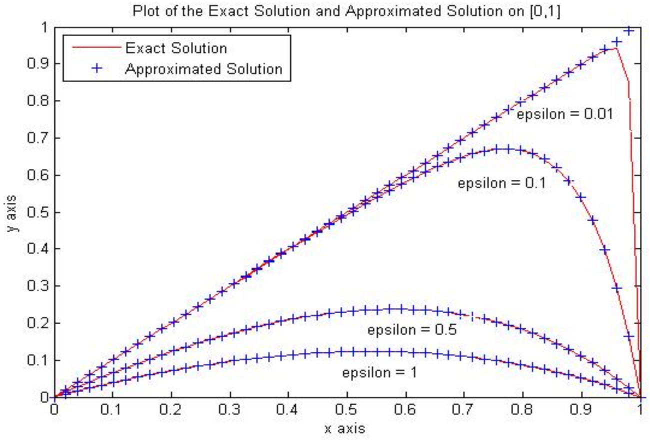

Figure: $n = 50$, $\epsilon = 1, 0.5, 0.1$ and 1

Numerical Results (Cont.)

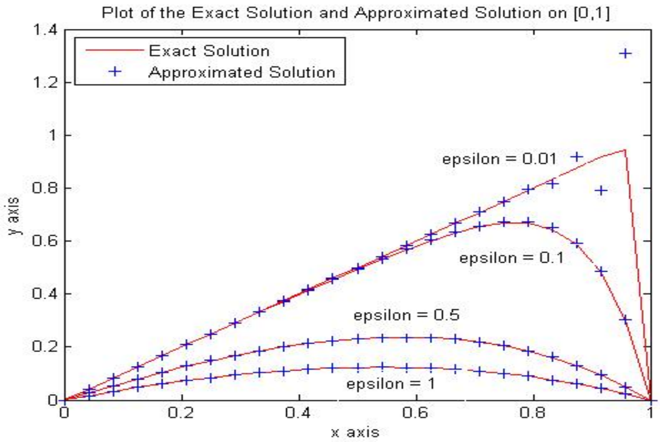

Figure: $n = 25$, $\epsilon = 1, 0.5, 0.1$ and 1

Numerical Results (Cont.)

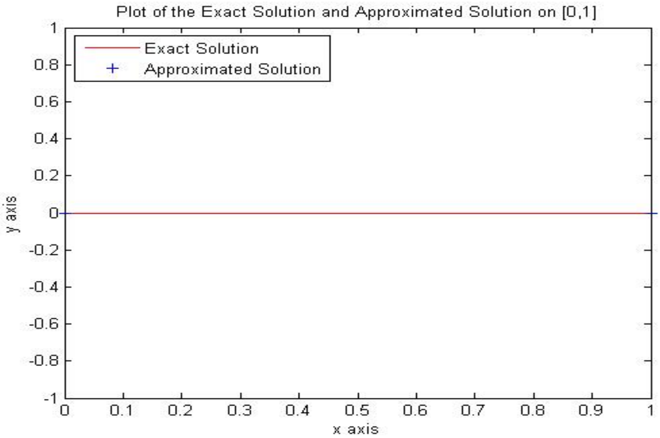

Figure: $n = 2$, $\epsilon = 1, 0.5, 0.1$ and 1

Discussion of Results

In order to explain the figures obtained on the previous slides, we perform some analysis on the behavior of the centered-difference approximation for the Advection-Diffusion Problem in 1D.
We rewrite Equation 11 as a difference equation in the form

$$aU_{i+1} + bU_i + cU_{i-1} = 1, \quad (i \geq 1) \tag{14}$$

where, $a = \frac{1}{2h} - \frac{\epsilon}{h^2}$, $b = \frac{2\epsilon}{h^2}$ and $c = \frac{-1}{2h} - \frac{\epsilon}{h^2}$

Discussion of Results (Cont.)

The analysis is performed on the homogeneous solution of our difference equation (Equation 14). To find the homogeneous solution, we assume a trial solution $U_i = x^i$. Substituting $U_i = x^i$, $U_{i+1} = x^{i+1}$ and $U_{i-1} = x^{i-1}$ into the homogeneous part of Equation 14 gives

$$ax^{i+1} + bx^i + cx^{i-1} = 0$$
$$\implies ax^2 + bx + c = 0$$

which has solution,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{15}$$

Discussion of Results (Cont.)

Plugging the values of a, b and c into Equation 15, gives

$$x_1 = 1 \quad \text{and} \quad x_2 = \frac{-2\epsilon - h}{-2\epsilon + h} \tag{16}$$

Thus the complimentary solution is

$$U_h = C_1 + C_2\left(\frac{-2\epsilon - h}{-2\epsilon + h}\right)^i, (i \geq 1) \tag{17}$$

Discussion of Results (Cont.)

The solution obtained suggest that, if $\epsilon > 0.01$, the approximate solution is consistent with the exact solution.

However, if $\epsilon \leq 0.01$ the approximate solution oscillates. This is because, for $\epsilon \leq 0.01$, x_2 in Equation 16 is negative.

Conclusions

In this project, we discussed the centered-difference method for the Advection-Diffusion problem in 1D.

We analyzed the approximated solution U_h and we concluded that this method performs well for large values of ϵ. However, it fails to approximate the solution for small values of ϵ.

We presented some analytical behavior of the problem which explains the presence of oscillations in the approximated solution for small values of ϵ.

Thank You!