Optimal Control for a Discrete Time Influenza Model

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Outline

- Continuous time models have been used to study influenza outbreaks and the impact of different control policies.
- The identification of optimal control strategies that involve antiviral treatment and isolation have been investigated in the continuous case.
- We formulate a discrete time model and we include social distancing and antiviral treatment as control policies.

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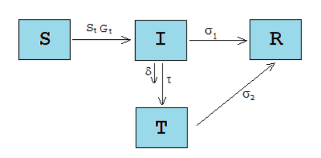
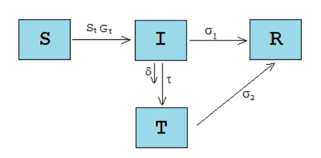


- The model is given by the system of difference equations:

$$\begin{aligned} S_{t+1} &= S_t(1-G_t) \\ I_{t+1} &= S_t G_t + (1-\tau_t)(1-\sigma_1)(1-\delta) I_t \\ T_{t+1} &= (1-\sigma_2)\,T_t + \tau_t(1-\sigma_1)(1-\delta) I_t \\ R_{t+1} &= R_t + \sigma_1(1-\delta)\,I_t + \sigma_2 T_t \end{aligned} \quad (1)$$

- Where

$$G_t = \rho(1-x_t)\frac{I_t + \epsilon T_t}{N_t}, \quad (2)$$



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- Our goal is to minimize the number of infected individuals over a finite interval $[0, n]$ by using a minimal effort on treatment and social distancing.
- The problem can be written as

$$\min \frac{1}{2} \sum_{t=0}^{n-1} \left(B_1 I_t^2 + B_2 x_t^2 + B_3 \tau_t^2 \right), \tag{3}$$

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- The Hamiltonian at time t is defined as

$$H_t = \frac{1}{2}\left(B_1 I_t^2 + B_2 x_t^2 + B_3 \tau_t^2\right) + \lambda_{t+1}^i y_{t+1}^i.$$

- The necessary conditions are given by:
  - The adjoint equation

$$\lambda_t^i = \frac{\partial H_t}{\partial y_t^i},$$

  - The transversality condition

$$\lambda_n^i = 0,$$

  - The optimality condition

$$\frac{\partial H_t}{\partial x_t} = 0 \quad \text{and} \quad \frac{\partial H_t}{\partial \tau_t} = 0.$$

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Interior-Point Methods (IPM)

- IPM were introduced by Karmarkar in 1984 for solving linear programming problems.
- Contrary to the simplex method, IPM find an optimal solution by crossing the interior of the feasible region.
- Problem (3) is posed as a nonlinear programming problem,

$$\begin{array}{ll} \min & f(y) \\ \text{s.t} & E(y) = 0, \\ & y \geq 0 \end{array} \tag{4}$$

where $\mathbf{y} = (S_1, I_1, T_1, x_0, \tau_0, \ldots, S_n, I_n, T_n, x_{n-1}, \tau_{n-1})$, and

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- The Lagrangian associated to (4) is

$$L(y, w, z) = f(y) + E(y)^T w - y^T z.$$

- The perturbed KKT conditions associated with (4) can be written as

$$F = \begin{bmatrix} \nabla_y L(y, w, z) \\ E(y) \\ YZ - \mu e \end{bmatrix} = 0 \qquad (6)$$

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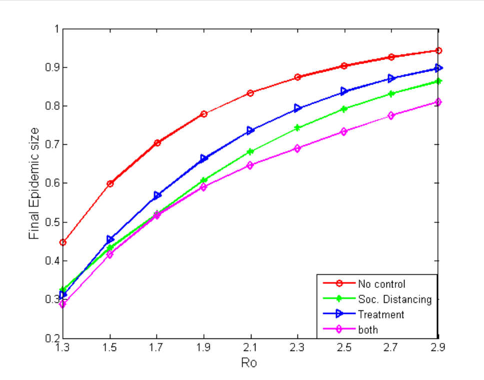
Table : Comparison between Forward-Backward and Interior-Point methods.

	FB		IPM	
Strategy	# of iterations	F	# of iterations	F
1	52	0.67997	11	0.68069
2	54	0.33014	10	0.33018
3	87	0.30423	23	0.30092

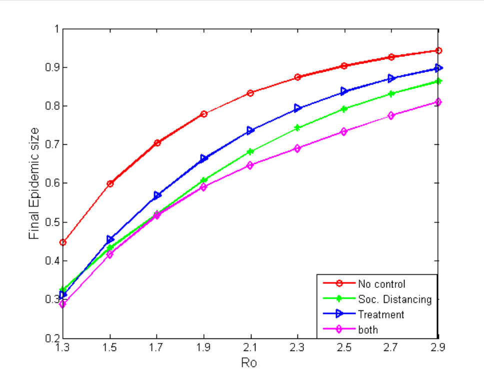
For all strategies IPM reach the solution with fewer number of iterations.

Final epidemic size vs. R_0



- We compare the final epidemic size for different values of R_0 under each strategy.
- Strategy 3 yields the highest reduction of the final epidemic size.
- For single policies Strategy 1 has more impact in the reduction of the final epidemic size than Strategy 2

Final epidemic size vs. R_0



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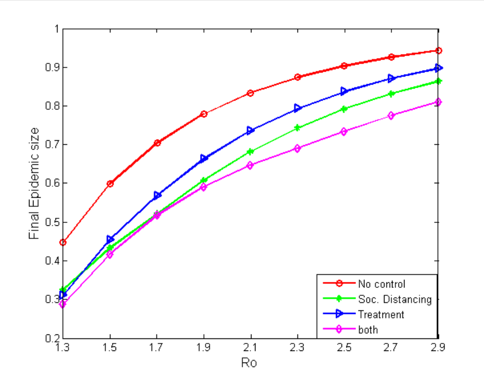

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- An isoperimetric constraint is included

$$\sum_{t=0}^{n-1} (\tau_i(t) I_i(t)) = k, \tag{7}$$

where k represents the available number of treatment doses.
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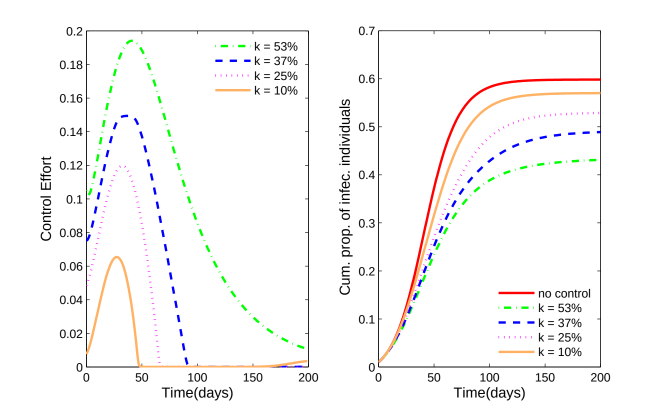
Numerical Results, $R_0 = 1.5$



For small values of k, the optimal control solution requires the implementation of highest values of treatment at the beginning of the epidemic until the resources are expended.

- We explore the role of heterogeneity via a discrete time epidemiological model involving interacting groups.
- The total population is divided into m subgroups according to to the contact activity or susceptibility levels.
- The fraction of susceptible individuals on group i at time t that get infected at time $t+1$ is given by

$$G_i = \rho_i \sum_{j=1}^{m} \left(q_j (1 - x_j(t)) \left(\frac{I_j(t) + \epsilon_j T_j(t)}{N_j} \right) \right), \tag{8}$$

where q_i (proportionate mixing) is given by

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- If n denotes the final time, the optimal control problem is formulated as

$$\min \frac{1}{2}\sum_{t=0}^{n-1}\sum_{i=1}^{m}\left(B_{I_i} I_i(t)^2 + B_{x_i} x_i(t)^2 + + B_{\tau_i}\tau_i(t)^2\right) \tag{11}$$

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- According to the USA Census (27%) of the population is in Group 1, (80%) in Group 2, and (13%) in Group 3.
- The biggest effort has to be applied both in the highest activity level Group 1 and the larger population size Group 2.
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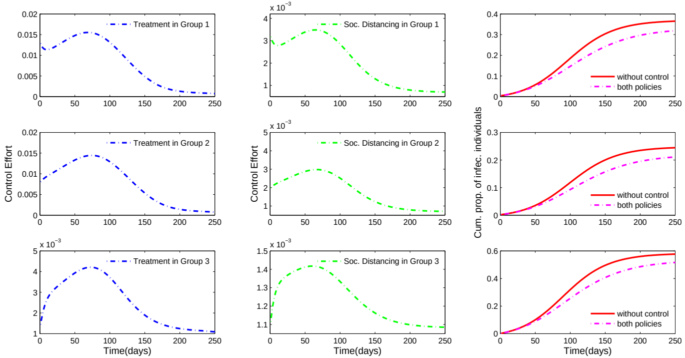

- The biggest effort both in treatment and social distancing has to be implemented in the highest risk Group 2.
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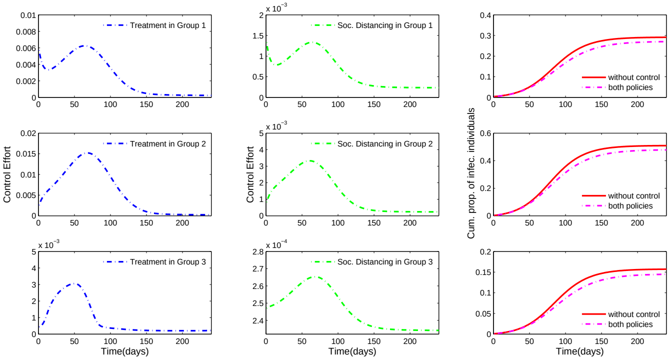

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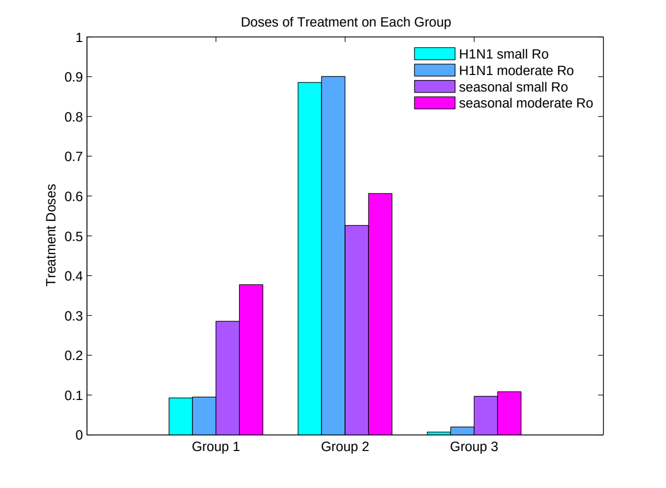

Conclusions

- An optimal control problem is solved by using two techniques: Forward-Backward algorithm and interior-point methods.
- Interior-point methods reach the solution with fewer number of iterations.
- IPM allows to incorporate inequality constraints and the isoperimetric constraint in a more efficient way.
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- In the case of limited resources, the maximum effort in control have to be implemented at the beginning of the epidemic.

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Related Publications

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A note on the use of optimal control on a discrete time model of influenza dynamics.
Math. Biosc. & Eng. 8, 183–197.

E. Fenichel, C. Castillo-Chavez, G. Ceddia, G. Chowell, P. Gonzalez-Parra, L. Velazquez et al. (2011)
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