Interior-Point Methods for an Optimal Control Influenza Model

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Outline

- Different continuous time approaches have been used to study single influenza outbreaks.
- The identification of optimal control strategies that involve antiviral treatment and isolation have been investigated in the continuous case.
- We formulate a discrete time model and we include social distancing and treatment as control policies.

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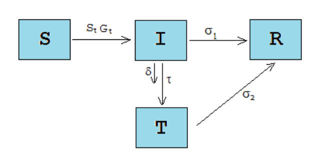
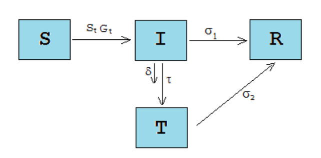


- The model is given by the system of difference equations:

$$\begin{aligned} S_{t+1} &= S_t(1-G_t) \\ I_{t+1} &= S_tG_t + (1-\tau_t)(1-\sigma_1)(1-\delta)I_t \\ T_{t+1} &= (1-\sigma_2)\,T_t + \tau_t(1-\sigma_1)(1-\delta)I_t \\ R_{t+1} &= R_t + \sigma_1(1-\delta)\,I_t + \sigma_2 T_t \end{aligned} \tag{1}$$

- Where

$$G_t = \rho\gamma(1-x_t)\frac{I_t + \epsilon T_t}{N_t}, \tag{2}$$



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- Our goal is to minimize the number of infected individuals over a finite interval $[0, n]$ by using a minimal effort on treatment and social distancing.
- The problem can be written as

$$\min \frac{1}{2} \sum_{t=0}^{n-1} \left(B_1 I_t^2 + B_2 x_t^2 + B_3 \tau_t^2 \right), \tag{3}$$

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- The Hamiltonian at time t is defined as

$$H_t = \frac{1}{2}\left(B_1 I_t^2 + B_2 x_t^2 + B_3 \tau_t^2\right) + \lambda_{t+1}^i y_{t+1}^i.$$

- The necessary conditions are given by:
  - The adjoint equation

$$\lambda_t^i = \frac{\partial H_t}{\partial y_t^i},$$

  - The transversality condition

$$\lambda_n^i = 0,$$

  - The optimality condition

$$\frac{\partial H_t}{\partial x_t} = 0 \quad \text{and} \quad \frac{\partial H_t}{\partial \tau_t} = 0.$$

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- IPM were introduced by Karmarkar in 1984 for solving linear programming problems.
- Contrary to the simplex method, IPM find an optimal solution by crossing the interior of the feasible region.
- Problem (3) is posed as a nonlinear programming problem,

$$\begin{aligned} \min \quad & f(y) \\ \text{s.t} \quad & E(y) = 0, \\ & y \geq 0 \end{aligned} \tag{4}$$

where $\mathbf{y} = (S_1, I_1, T_1, x_0, \tau_0, \ldots, S_n, I_n, T_n, x_{n-1}, \tau_{n-1})$, and

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- The Lagrangian associated to (4) is

$$L(y, w, z) = f(y) + E(y)^T w - y^T z.$$

- The perturbed KKT conditions associated with (4) can be written as

$$F = \begin{bmatrix} \nabla_y L(y, w, z) \\ E(y) \\ YZ - \mu e \end{bmatrix} = 0 \qquad (6)$$

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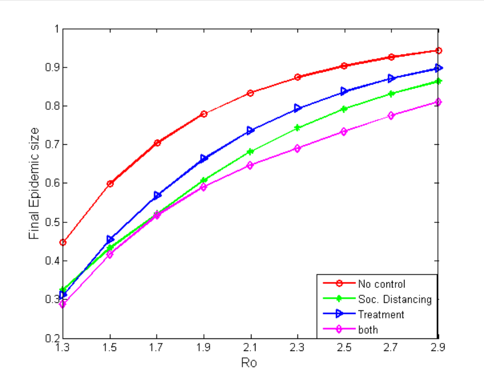
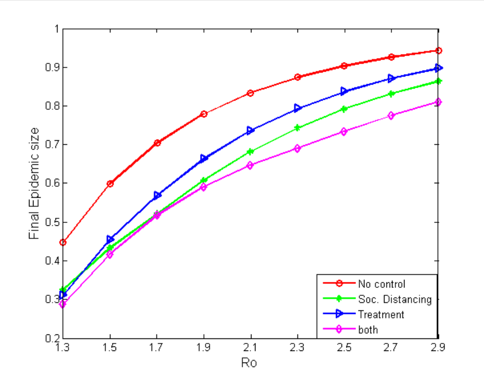
Table: Comparison between Forward-Backward and Interior-Point methods.

	FB		IPM	
Strategy	# of iterations	F	# of iterations	F
1	52	0.67997	11	0.68069
2	54	0.33014	10	0.33018
3	87	0.30423	23	0.30092

For all strategies IPM reach the solution with fewer number of iterations.



- We compare the final epidemic size for different values of R_0 under each strategy.
- Strategy 3 yields the highest reduction of the final epidemic size.
- For single policies Strategy 1 has more impact in the reduction of the final epidemic size than Strategy 2



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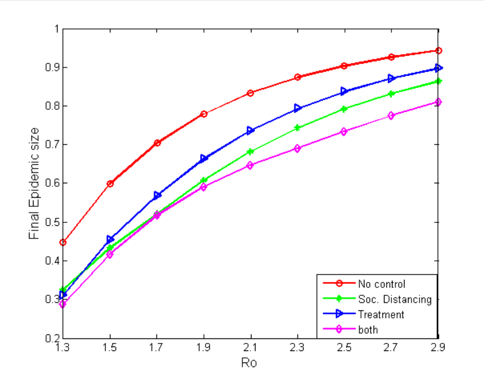

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- We want to consider the case of limited resources.
- An isoperimetric constraint is included

$$\sum_{t=0}^{n-1} (\tau_i(t) I_i(t)) = k, \tag{7}$$

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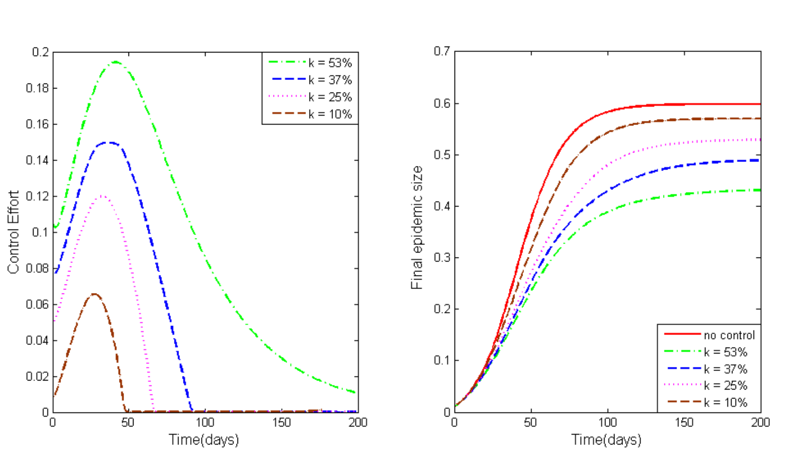
Numerical Results, $R_0 = 1.5$


Control Effort
Time(days)
k = 53%
k = 37%
k = 25%
k = 10%
Final epidemic size
Time(days)
no control
k = 53%
k = 37%
k = 25%
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Numerical Results, $R_0 = 2.6$

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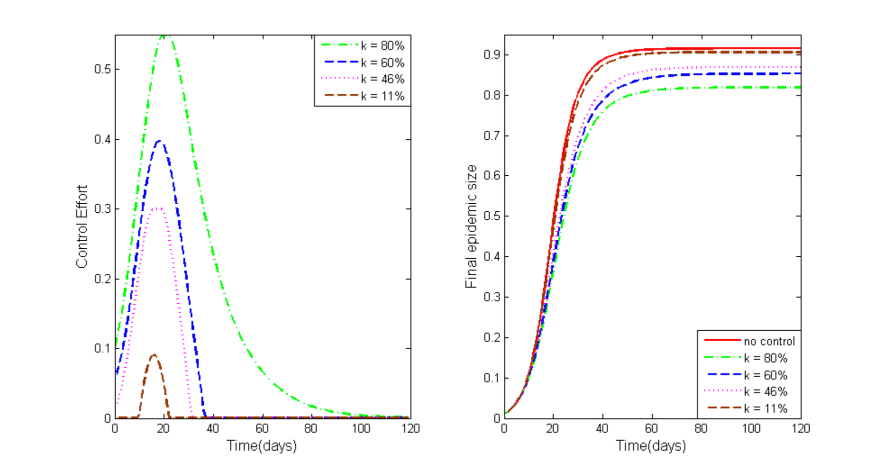

- An optimal control problem is solved by using two techniques: Forward-Backward algorithm and interior-point methods.
- Interior-point methods reach the solution with fewer number of iterations.
- The use of single and dual strategies results in reductions in the cumulative number of infected individuals.
- Under the implementation of a single policy, the social distancing strategy is more effective than the antiviral treatment when $R_0 > 1.4$.
- Future Work: Extend the model to an age-structured model and analize how control policies should be implemented in each group.

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A note on the use of optimal control on a discrete time model of influenza dynamics.
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Adaptive human behavior in epidemiological models.
Proceedings of the National Academy of Sciences, 108(15).

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