Chapter 12
Intro to Waiting Line Models

- The Structure of a Waiting Line System
- Queuing Systems
- Queuing System Input Characteristics
- Queuing System Operating Characteristics
- Analytical Formulas
- The Single-Channel Waiting Line Model with Poisson Arrivals and Exponential Service Times
- The Multiple-Channel Waiting Line Model with Poisson Arrivals and Exponential Service Times
- Economic Analysis of Waiting Lines

Structure of a Waiting Line System

- Queuing theory is the study of waiting lines.
- Four characteristics of a queuing system are:
  - the manner in which customers arrive
  - the time required for service
  - the priority determining the order of service
  - the number and configuration of servers in the system.
Components of the Queuing Phenomenon
Servicing System

Structure of a Waiting Line System

- In general, the arrival of customers into the system is a random event.
- Frequently the arrival pattern is modeled as a Poisson process.
- Service time is also usually a random variable.
- A distribution commonly used to describe service time is the exponential distribution.
- The most common queue discipline is first come, first served (FCFS).
- An elevator is an example of last come, first served (LCFS) queue discipline.
Queuing Systems

- A three part code of the form \( A/B/s \) is used to describe various queuing systems.
- \( A \) identifies the arrival distribution, \( B \) the service (departure) distribution and \( s \) the number of servers for the system.
- Frequently used symbols for the arrival and service processes are: \( M \) - Markov distributions (Poisson/exponential), \( D \) - Deterministic (constant) and \( G \) - General distribution (with a known mean and variance).
- For example, \( M/M/k \) refers to a system in which arrivals occur according to a Poisson distribution, service times follow an exponential distribution and there are \( k \) servers working at identical service rates.

Poisson Arrivals

- Customer arrivals are independent of each other
- The probability that a customer will arrive within a time interval is proportional to the length of the interval
- The probability that two or more arrivals occur at the same time is practically zero
“Poisson Arrivals”

- $\lambda =$ average number of arrivals per time period

$$P(X = x) = p(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \ x = 0, 1, 2, \ldots$$

“Exponential Service Times”

- $\mu =$ average number of units served per time period

$$P(\text{service time } \leq t) = 1 - e^{-\mu t}$$
Queuing System Input Characteristics

\( \lambda = \) the average arrival rate

\( \frac{1}{\lambda} = \) the average time between arrivals ("interarrival time")

\( \mu = \) the average service rate for each server

\( \frac{1}{\mu} = \) the average service time

\( \sigma = \) the standard deviation of the service time

---

Queuing System Operating Characteristics

\( P_0 = \) probability the service facility is idle

\( P_n = \) probability of \( n \) units in the system

\( P_w = \) probability an arriving unit must wait for service

\( L_q = \) average number of units in the queue awaiting service

\( L = \) average number of units in the system

\( W_q = \) average time a unit spends in the queue awaiting service

\( W = \) average time a unit spends in the system
Analytical Formulas

- For nearly all queuing systems, there is a relationship between the average time a unit spends in the system or queue and the average number of units in the system or queue. These relationships, known as Little's flow equations are:
  \[ L = \lambda W \quad \text{and} \quad L_q = \lambda W_q \]

- When the queue discipline is FCFS, analytical formulas have been derived for several different queuing models including the following: \( M/M/1 \), \( M/M/k \), \( M/G/1 \), \( M/G/k \) with blocked customers cleared, and \( M/M/1 \) with a finite calling population.

- Analytical formulas are not available for all possible queuing systems. In this event, insights may be gained through a simulation of the system.