Course:	MATH 2315 Calculus III
Class:	Study Guide 2
Semester:	Spring 2022
Related Exam:	Midterm Exam 2
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Objectives

Reinforce the concepts of: Velocity and Acceleration, Tangent Vectors and Normal Vectors, Arc Length and Curvature, Functions of Several Variables, Limits and Continuity, Partial Derivatives, Differentials, Chain Rules for Functions of Several Variables, Directional Derivatives and Gradients, and Tangent Planes and Normal Lines.

1. A projectile is fired from a gun at an initial height of 2 meters above the ground. The bullet has a muzzle velocity of 120 meters per second, and the barrel of the gun is angled at 45° above the horizontal. Find the range of the projectile, and its maximum height.

Explanation

i. Consider the function $r(t) = (120\cos 45°)\,t\,\mathbf{i} + \left[2 + (120\sin 45°)\,t - \frac{1}{2}gt^2\right]\mathbf{j}$ and then find the horizontal position when the height equals zero.

ii. Consider $y'(t) = 0$ and find the corresponding value of $y(t)$.

2. A B-2 bomber releases a bomb at an altitude of 10,000 meters as it travels at 900 kilometers per hour. Find the normal and tangential components of acceleration that act on the bomb.

Explanation

Consider the function $r(t) = 250t\,\mathbf{i} + \left(10000 - \frac{1}{2}gt^2\right)\mathbf{j}$ and then find $a_T = \dfrac{v \cdot a}{||v||}$ and $a_N = \dfrac{||v \times a||}{||v||}$.

3. Given the curve C described by $y = 2/x$, find P, where P is the point on C where the curvature K reaches its maximum value. Find $\lim_{x\to\infty} K$.

Explanation

Find $K(x) = \dfrac{|y''|}{\left[1 + (y')^2\right]^{3/2}}$ and then determine x such that $dK/dx = 0$.

4. A wooden box measures l cm in length, w cm in width, and h cm in height. The wood used to build the box costs \$0.5 per square meter for the base and the top, and \$0.3 per square meter for the sides. Write the cost C of building the wooden box as a function of l, w, and h.

Explanation

Consider the dimensions of the six sides of the box and define the function $C(l, w, h)$.

5. Given the function $g(x, y)$, define $f(0, 0)$ such that the function is continuous at $(0, 0)$.

$$g(x, y) = 2xy\frac{3x^2 - y^2}{x^2 + y^2}$$

Explanation

Consider the change to polar coordinates:

$$\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{r\to0} 2r\cos\theta\, r\sin\theta\frac{3r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

6. A metal sheet has temperature $T(x, y) = 400 - 0.5x^2 - 1.4y^2$, in °C, at any point (x, y) on its surface, where x and y are measured in centimeters. Find the rates of change of $T(x, y)$ at $(1, 2)$ with respect to the distances moved along the sheet in the directions of the coordinate axes.

Explanation

Calculate $\frac{\partial T}{\partial x}(1, 2)$ and $\frac{\partial T}{\partial y}(1, 2)$.

7. The function $T(g, L) = 2\pi\sqrt{L/g}$ describes the period of a pendulum, where L is the length of the pendulum, and g is the acceleration due to gravity at the point where the pendulum is located. A pendulum located in Quito, Ecuador, where $g = 9.775$ m/s^2, is moved to Murmansk, Russia, where $g = 9.825$ m/s^2. The length of the pendulum is reduced from 80 cm to 75 cm due to the difference in temperature between both cities. Approximate the change in $T(g, L)$.

Explanation

Assume that Δg and ΔL are sufficiently small to use the approximation:

$$\Delta T \approx dT = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)dx + f_y(x, y)dy$$

8. The Ideal Gas Law is the equation of state of an ideal gas (i.e. a hypothetical gas formed by point particles that to not interact with each other), and is written as $PV = nRT$, where T is the temperature of the gas, V is the volume it occupies, P is its pressure, R is the ideal gas constant, and n is the amount of substance. Considering that P and V are functions of time, find dT/dt.

Explanation

Find T in terms of P and V and use the Chain Rule for one independent variable:

$$\frac{dT}{dt} = \frac{\partial T}{\partial P}\frac{dP}{dt} + \frac{\partial T}{\partial V}\frac{dV}{dt}$$

9. The surface of Mount Rainier is described by the equation $f(x, y) = 4392 - 0.005x^2 - 0.003y^2$. A mountain climber stands at $P(400, 200, 3472)$. If they wanted to ascend at the greatest possible rate, in which direction should they move?

Explanation

Calculate the gradient of f at $(400, 200)$ to obtain the direction of maximum increase of $f(x, y)$.

10. Find a point on the surface S, given by $x^2 + 2y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $x + 2y - z = 0$.

Explanation

i. Consider $F(x, y, z) = x^2 + 2y^2 - z^2 - 1$, then $\nabla F(x, y, z)$ is normal to plane that is tangent to S at (x, y, z).

ii. $\nabla F(x, y, z)$ is thus parallel to the plane's normal vector. That means $\nabla F(x, y, z) = t\langle 1, 2, -1\rangle$ for some t.

iii. Find x, y, and z in terms of t and substitute into the equation of S.

iv. Solve for t and calculate the coordinates of the point.

Answer Key:

1. Range ≈ 1471.39 m , Maximum height ≈ 369.35 m

2. $a_T = \dfrac{9.8^2 t}{\sqrt{250^2 + 9.8^2 t^2}} \dfrac{m}{s^2}$, $a_N = \dfrac{2540}{\sqrt{250^2 + 9.8^2 t^2}} \dfrac{m}{s^2}$

3. There are two points: $P_1(\sqrt{2}, \sqrt{2})$, $P_2(-\sqrt{2}, -\sqrt{2})$; $\lim_{x\to\infty} K = 0$

4. $C(l, w, h) = lw + 0.6(lh + wh)$

5. $g(x, y) = 0$

6. $\dfrac{\partial T}{\partial x}(1, 2) = -1 \dfrac{°C}{cm}$, $\dfrac{\partial T}{\partial y}(1, 2) = -5.6 \dfrac{°C}{cm}$

7. $\Delta \approx -0.179$ seconds

8. $\dfrac{dT}{dt} = \dfrac{1}{nR}\left(V\dfrac{dP}{dt} + P\dfrac{dV}{dt}\right)$

9. $\nabla f(400, 200) = -4\ \mathbf{i} - 1.2\ \mathbf{j}$

10. There are two points: $P_1\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$, $P_2\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$

Cheat Sheet

Position, velocity, acceleration, and curvature

Position vector

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$
$$= [x_0 + (v_0 \cos\theta)\,t]\,\mathbf{i} + \left[y_0 + (v_0 \sin\theta)\,t - \tfrac{1}{2}gt^2\right]\mathbf{j}$$

Velocity vector

$$v(t) = \mathbf{r}'(t)$$

Speed

$$||v(t)|| = ||\mathbf{r}'(t)||$$

Acceleration vector

$$\mathbf{a}(t) = \mathbf{r}''(t) = a_T\mathbf{T}(t) + a_N\mathbf{N}(t)$$

Unit tangent vector

$$T(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$$

Principal unit normal vector

$$N(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$$

Tangential component of acceleration

$$a_T = \mathrm{a} \cdot \mathbf{T} = \frac{\mathrm{v} \cdot \mathrm{a}}{||\mathrm{v}||}$$

Normal component of acceleration

$$a_N = \mathrm{a} \cdot \mathbf{N}$$
$$= \frac{||\mathrm{v} \times \mathrm{a}||}{||\mathrm{v}||}$$
$$= \sqrt{||\mathrm{a}||^2 - a_T{}^2}$$

Curvature in the plane

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}$$
$$= \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}}$$

Curvature in the plane or in space

$$K = ||\mathbf{T}'(s)|| = ||\mathbf{r}''(s)||$$
$$= \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$$
$$= \frac{\mathrm{a}(t) \cdot \mathbf{N}(t)}{||\mathrm{v}(t)||^2}$$

Functions of several variables

Continuity of a function of two variables

The function $f(x, y)$ is continuous at (x_0, y_0) in an open region R if $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Total differential

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then their differentials are $dx = \Delta x$ and $dy = \Delta y$, and the total differential of the dependent variable is $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)dx + f_y(x, y)dy$.

Chain rule: one independent variable

Let $w = f(x, y)$, where f is differentiable on x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable on t, then $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$.

Chain rule: two independent variables

If $w = f(x, y)$, where f is differentiable on x and y, and $x = g(s, t)$ and $y = h(s, t)$, such that $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist. Then $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$, and $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$.

Directional derivative

If $f(x, y)$ is differentiable on x and y, then the derivative of f in the direction of $u = \cos\theta\, i + \sin\theta\, j$ is $D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$.

Gradient of a function of two variables

Let $z = f(x, y)$, such that f_x and f_y exist. Then the gradient of f, denoted by $\nabla f(x, y)$ is given by:

$$\nabla f(x, y) = f_x(x, y)\,i + f_y(x, y)\,j$$

Equation of tangent plane

If F is differentiable at (x_0, y_0, z_0), then the equation of the tangent plane to the surface $F(x, y, z) = 0$ at (x_0, y_0, z_0) is:

$$0 = F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)$$